Laplace Transforms
Of
Differential Expressions

By

J. Edward Oxford

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Theorem

Given a differential expression:

$$\text{D: } a_1 y^{(n)} + a_2 y^{(n-1)} + a_3 y^{(n-2)} + \cdots + a_n y' + a_{n+1} y$$

The Laplace Transform of D is as follows:

$$L\{\text{D}\} = F(S)[a_1 S^n + a_2 S^{n-1} + \cdots + a_n S + a_{n+1}] - y(0)[a_1 S^{n-1} + a_2 S^{n-2} + \cdots + a_{n-1} S + a_n] - y'(0)[a_1 S^{n-2} + a_2 S^{n-3} + \cdots + a_{n-2} S + a_{n-1}] - \cdots - y^{n-2}(0)[a_1 S - a_2] - y^{n-1}(0)[a_1]$$

Where,
$F(S) = L\{y(t)\}$
$a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n, a_{n+1}$ are constant coefficients.

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Proof

Preface:

This proof contains two sections. The first section outlines the proof in general terms with implied continuation of the pattern throughout. The second section aims to demonstrate the accuracy of the first section by showing a general proof without implied continuation of a pattern and with finite, 'n', terms.

Section 1

Theorem:

The Laplace Transform of a higher order derivative of order 'n' is:

① $L\{y^{(n)}(t)\} = S^n L\{y(t)\} - S^{n-1}y(0) - S^{n-2}y'(0) - \cdots - y^{n-1}(0)$

Zill p.284

Given the following differential expression:

$$\text{D: } a_1 y^{(n)} + a_2 y^{(n-1)} + a_3 y^{(n-2)} + \cdots + a_{n-1}y'' + a_n y' + a_{n+1}y$$

Where $a_1, a_2, a_3, a_4, \ldots, a_{n-1}, a_n, a_{n+1}$ are constant coefficients.

We wish to obtain $L\{D\}$.

$$\therefore L\{D\} = L\{a_1 y^{(n)} + a_2 y^{(n-1)} + a_3 y^{(n-2)} + \cdots + a_{n-1}y'' + a_n y' + a_{n+1}y\}$$

$$\Rightarrow L\{D\} = L\{a_1 y^{(n)}\} + L\{a_2 y^{(n-1)}\} + L\{a_3 y^{(n-2)}\} + \cdots + L\{a_{n-1}y''\} + L\{a_n y'\} + L\{a_{n+1}y\}$$

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Using ① we get the following:

$$L\{a_1 y^{(n)}\} = a_1[S^n L\{y(t)\} - S^{n-1}y(0) - S^{n-2}y'(0) - \cdots - Sy^{n-2}(0) - y^{n-1}(0)]$$

$$L\{a_2 y^{(n-1)}\} = a_2\left[S^{(n-1)} L\{y(t)\} - S^{(n-1)-1}y(0) - S^{(n-1)-2}y'(0) - \cdots - y^{(n-1)-1}(0)\right]$$

$$L\{a_3 y^{(n-2)}\} = a_3\left[S^{(n-2)} L\{y(t)\} - S^{(n-2)-1}y(0) - S^{(n-2)-2}y'(0) - \cdots - y^{(n-2)-1}(0)\right]$$

$$\vdots$$

$$L\{a_{n-1}y''\} = a_{n-1}[S^2 L\{y(t)\} - Sy(0) - y'(0)]$$

$$L\{a_n y'\} = a_n[SL\{y(t)\} - y(0)]$$

$$L\{a_{n+1}y\} = a_{n+1}[L\{y(t)\}]$$

Now,

$$L\{D\} = L\{a_1 y^{(n)}\} + L\{a_2 y^{(n-1)}\} + L\{a_3 y^{(n-2)}\} + \cdots + L\{a_{n-1}y''\} + L\{a_n y'\} + L\{a_{n+1}y\}$$

$$\begin{aligned}\therefore L\{D\} = {} & a_1[S^n L\{y(t)\} - S^{n-1}y(0) - S^{n-2}y'(0) - \cdots - Sy^{n-2}(0) - y^{n-1}(0)] \\ & + a_2\left[S^{(n-1)} L\{y(t)\} - S^{(n-1)-1}y(0) - S^{(n-1)-2}y'(0) - \cdots - y^{(n-1)-1}(0)\right] \\ & + a_3\left[S^{(n-2)} L\{y(t)\} - S^{(n-2)-1}y(0) - S^{(n-2)-2}y'(0) - \cdots - y^{(n-2)-1}(0)\right] + \cdots \\ & + a_{n-1}[S^2 L\{y(t)\} - Sy(0) - y'(0)] + a_n[SL\{y(t)\} - y(0)] + a_{n+1}[L\{y(t)\}]\end{aligned}$$

Cleaning up the exponents we get:

$$\begin{aligned}L\{D\} = {} & a_1[S^n L\{y(t)\} - S^{n-1}y(0) - S^{n-2}y'(0) - \cdots - Sy^{n-2}(0) - y^{n-1}(0)] \\ & + a_2\left[S^{(n-1)} L\{y(t)\} - S^{(n-2)}y(0) - S^{(n-3)}y'(0) - \cdots - y^{(n-2)}(0)\right] \\ & + a_3\left[S^{(n-2)} L\{y(t)\} - S^{(n-3)}y(0) - S^{(n-4)}y'(0) - \cdots - y^{(n-3)}(0)\right] + \cdots \\ & + a_{n-1}[S^2 L\{y(t)\} - Sy(0) - y'(0)] + a_n[SL\{y(t)\} - y(0)] + a_{n+1}[L\{y(t)\}]\end{aligned}$$

Now, let $L\{y(t)\} = F(S)$.

$$\begin{aligned}L\{D\} = {} & a_1[S^n F(S) - S^{n-1}y(0) - S^{n-2}y'(0) - \cdots - Sy^{n-2}(0) - y^{n-1}(0)] \\ & + a_2\left[S^{(n-1)} F(S) - S^{(n-2)}y(0) - S^{(n-3)}y'(0) - \cdots - y^{(n-2)}(0)\right] \\ & + a_3\left[S^{(n-2)} F(S) - S^{(n-3)}y(0) - S^{(n-4)}y'(0) - \cdots - y^{(n-3)}(0)\right] + \cdots \\ & + a_{n-1}[S^2 F(S) - Sy(0) - y'(0)] + a_n[SF(S) - y(0)] + a_{n+1}[F(S)]\end{aligned}$$

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Now, let us begin to clean up the previous expression by multiplying through by the appropriate coefficients $a_1, a_2, a_3, a_4, \dots, a_{n-1}, a_n, a_{n+1}$:

$$\begin{aligned} L\{D\} = {} & [a_1 S^n F(S) - a_1 S^{n-1} y(0) - a_1 S^{n-2} y'(0) - \cdots - a_1 S y^{n-2}(0) - a_1 y^{n-1}(0)] \\ & + [a_2 S^{(n-1)} F(S) - a_2 S^{(n-2)} y(0) - a_2 S^{(n-3)} y'(0) - \cdots - a_2 y^{(n-2)}(0)] \\ & + [a_3 S^{(n-2)} F(S) - a_3 S^{(n-3)} y(0) - a_3 S^{(n-4)} y'(0) - \cdots - a_3 y^{(n-3)}(0)] + \cdots \\ & + [a_{n-1} S^2 F(S) - a_{n-1} S y(0) - a_{n-1} y'(0)] + [a_n S F(S) - a_n y(0)] + [a_{n+1} F(S)] \end{aligned}$$

$$\begin{aligned} \therefore L\{D\} = {} & a_1 S^n F(S) - a_1 S^{n-1} y(0) - a_1 S^{n-2} y'(0) - \cdots - a_1 y^{n-1}(0) + a_2 S^{(n-1)} F(S) - a_2 S^{(n-2)} y(0) \\ & - a_2 S^{(n-3)} y'(0) - \cdots - a_2 y^{(n-2)}(0) + a_3 S^{(n-2)} F(S) - a_3 S^{(n-3)} y(0) - a_3 S^{(n-4)} y'(0) \\ & - \cdots - a_3 y^{(n-3)}(0) + \cdots + a_{n-1} S^2 F(S) - a_{n-1} S y(0) - a_{n-1} y'(0) + a_n S F(S) - a_n y(0) \\ & + a_{n+1} F(S) \end{aligned}$$

Now, combining like terms so as to arrange all $F(S)$ terms, the $y(0)$ terms, the $y'(0)$ terms, and so on in their respective groups, we get,

$$\begin{aligned} L\{D\} = {} & a_1 S^n F(S) + a_2 S^{(n-1)} F(S) + a_3 S^{(n-2)} F(S) + \cdots + a_{n-1} S^2 F(S) + a_n S F(S) + a_{n+1} F(S) \\ & - a_1 S^{n-1} y(0) - a_2 S^{(n-2)} y(0) - a_3 S^{(n-3)} y(0) - \cdots - a_{n-1} S y(0) - a_n y(0) \\ & - a_1 S^{n-2} y'(0) - a_2 S^{(n-3)} y'(0) - a_3 S^{(n-4)} y'(0) - \cdots - a_{n-1} y'(0) - a_1 S y^{n-2}(0) \\ & - a_2 y^{(n-2)}(0) - a_1 y^{n-1}(0) \end{aligned}$$

Now, factoring out $F(S)$,$\big(-y(0)\big)$, $\big(-y'(0)\big)$, and so on, we get the following:

$$\begin{aligned} L\{D\} = {} & F(S)[a_1 S^n + a_2 S^{n-1} + \cdots + a_n S + a_{n+1}] - y(0)[a_1 S^{n-1} + a_2 S^{n-2} + \cdots + a_{n-1} S + a_n] - \\ & y'(0)[a_1 S^{n-2} + a_2 S^{n-3} + \cdots + a_{n-2} S + a_{n-1}] - \cdots - y^{n-2}(0)[a_1 S - a_2] - y^{n-1}(0)[a_1] \end{aligned}$$

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Section 2

Let P_1 be an Ordinary Differential Expression of order 4 taking the form:

$$P_1: Ay^{(4)} + By''' + Cy'' + Dy' + Ey$$

Where,
A, B, C, D, and E are constant coefficients.

We wish to obtain the Laplace Transform of P_1.

$$L\{P_1\} = L\{Ay^{(4)} + By''' + Cy'' + Dy' + Ey\}$$

$$\therefore L\{P_1\} = L\{Ay^{(4)}\} + L\{By'''\} + L\{Cy''\} + L\{Dy'\} + L\{Ey\}$$

Using ① we get the following:

$$L\{Ay^{(4)}\} = A[S^4L\{y(t)\} - S^3y(0) - Sy^2(0) - Sy''(0) - y'''(0)]$$

$$L\{By'''\} = B[S^3L\{y(t)\} - S^2y(0) - Sy'(0) - y''(0)]$$

$$L\{Cy''\} = C[S^2L\{y(t)\} - Sy(0) - y'(0)]$$

$$L\{Dy'\} = D[SL\{y(t)\} - y(0)]$$

$$L\{Ey\} = E[L\{y(t)\}]$$

Now,

$$L\{P_1\} = L\{Ay^{(4)}\} + L\{By'''\} + L\{Cy''\} + L\{Dy'\} + L\{Ey\}$$

$$\therefore L\{P_1\} = A[S^4L\{y(t)\} - S^3y(0) - S^2y'(0) - Sy''(0) - y'''(0)] + B[S^3L\{y(t)\} - S^2y(0) - Sy'(0) - y''(0)] + C[S^2L\{y(t)\} - Sy(0) - y'(0)] + D[SL\{y(t)\} - y(0)] + E[L\{y(t)\}]$$

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Let $L\{y(t)\} = F(S)$, where F is a function of 'S'.

$$\therefore L\{P_1\} = A[S^4F(S) - S^3y(0) - S^2y'(0) - Sy''(0) - y'''(0)] + B[S^3F(S) - S^2y(0) - Sy'(0) - y''(0)] + C[S^2F(S) - Sy(0) - y'(0)] + D[SF(S) - y(0)] + E[F(S)]$$

If we clean up the above expression by multiplying through by the coefficients A, B, C, D, and E, and then grouping the $F(S)$ terms, the $y(0)$ terms, the $y'(0)$ terms, and so on, we get the following:

$$L\{P_1\} = (AS^4)F(S) + (BS^3)F(S) + (CS^2)F(S) + (DS)F(S) + (E)F(S) - AS^3y(0) - BS^2y(0) - CSy(0) - Dy(0) - AS^2y'(0) - BSy'(0) - Cy'(0) - ASy''(0) - By''(0) - Ay'''(0)$$

Now, factoring out $F(S)$,$\big(-y(0)\big)$, $\big(-y'(0)\big)$, and so on, we get the following:

$$L\{P_1\} = F(S)\underbrace{[(AS^4) + (BS^3) + (CS^2) + (DS) + (E)]}_{P_{1.1}} - y(0)[AS^3 + BS^2 + CS + D] - y'(0)[AS^2 + BS + C] - y''[AS + B] - y'''(0)[A]$$

Notice: The terms being multiplied by $F(S)$when factored yield an auxiliary equation to P_1using differential operators such that:

$$Ay^{(4)} + By''' + Cy'' + Dy' + Ey = \underbrace{(AD^4 + BD^3 + CD^2 + DD + E)}_{P_2}\, y$$

$P_{1.1}$takes the form of an auxiliary equation of P_2

Therefore we are left with the following:

② $L\{P_1\} = F(S)[(AS^4) + (BS^3) + (CS^2) + (DS) + (E)] - y(0)[AS^3 + BS^2 + CS + D] - y'(0)[AS^2 + BS + C] - y''[AS + B] - y'''(0)[A]$

This is the result obtained by applying the theorem derived strictly from the definition of a Laplace transform.

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Now,

$$P_1: Ay^{(4)} + By''' + Cy'' + Dy' + Ey$$

From my hypothesis:

$$L\{\mathrm{D}\} = F(S)[a_1S^n + a_2S^{n-1} + \cdots + a_nS + a_{n+1}] - y(0)[a_1S^{n-1} + a_2S^{n-2} + \cdots + a_{n-1}S + a_n] - y'(0)[a_1S^{n-2} + a_2S^{n-3} + \cdots + a_{n-2}S + a_{n-1}] - \cdots - y^{n-2}(0)[a_1S - a_2] - y^{n-1}[a_1]$$

$$\vdots$$

$$\therefore L\{P_1\} = F(S)[a_1S^n + a_2S^{n-1} + \cdots + a_nS + a_{n+1}] - y(0)[a_1S^{n-1} + a_2S^{n-2} + \cdots + a_{n-1}S + a_n] - y'(0)[a_1S^{n-2} + a_2S^{n-3} + \cdots + a_{n-2}S + a_{n-1}] - \cdots - y^{n-2}(0)[a_1S - a_2] - y^{n-1}[a_1]$$

Where,

$n = 4$ $\quad a_3 = a_{n-1} = C$

$a_1 = A$ $\quad a_4 = a_n = D$

$a_2 = a_{n-2} = B$ $\quad a_5 = a_{n+1} = E$

Plugging in we get:

③ $L\{P_1\} = F(S)[(AS^4) + (BS^3) + (CS^2) + (DS) + (E)] - y(0)[AS^3 + BS^2 + CS + D] - y'(0)[AS^2 + BS + C] - y''[AS + B] - y'''(0)[A]$

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Examples

Example 1.

$E_1 = 5y'' + 2y' + 8y$

From my hypothesis, we get:

$n = 2$	$a_2 = 2$
$a_1 = 5$	$a_3 = 8$

$\therefore L\{E_1\} = F(S)[a_1S^n + a_2S^{n-1} + a_1S^{n-2}] - y(0)[a_1S^{n-1} + a_2] - y^{(n-1)}(0)[a_1]$

$\Rightarrow L\{E_1\} = F(S)[5S^2 + 2S + 8] - y(0)[5S + 2] - y'(0)[5]$

Example 2.

$E_2 = y'' - 3y' + 2y \quad ; \quad y(0) = 1\,, y'(0) = 5$

From my hypothesis, we get:

$n = 2$	$a_1 = 1$
$y(0) = 1$	$a_2 = -3$
$y'(0) = 5$	$a_3 = 2$

$L\{E_2\} = F(S)[a_1S^n + a_2S^{n-1} + a_1S^{n-2}] - y(0)[a_1S^{n-1} + a_2S^{n-2}] - y^{(n-1)}(0)[a_1S^{n-2}]$

$\therefore L\{E_2\} = F(S)[S^2 - 3S + 2] - (1)[S - 3] - (5)[1]$

$\Rightarrow L\{E_2\} = F(S)[S^2 - 3S + 2] - S + 2$

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Example 3.

$E_3 = 7y^{(5)} + 13y^{(4)} + 19y''' + 5y'' + 2y' + 6y$

From my hypothesis, we get:

$n = 5$	$a_4 = 5$
$a_1 = 7$	$a_5 = 2$
$a_2 = 13$	$a_6 = 6$
$a_3 = 19$	

$$\therefore L\{E_3\} = F(S)[a_1S^n + a_2S^{n-1} + a_3S^{n-2} + a_4S^{n-3} + a_5S^{n-4} + a_6S^{n-5}]$$
$$- y(0)[a_1S^{n-1} + a_2S^{n-2} + a_3S^{n-3} + a_4S^{n-4} + a_5S^{n-5}]$$
$$- y'(0)[a_1S^{n-2} + a_2S^{n-3} + a_3S^{n-4} + a_4S^{n-5}] - y''(0)[a_1S^{n-3} + a_2S^{n-4} + a_3S^{n-5}]$$
$$- y'''(0)[a_1S^{n-4} + a_2S^{n-5}] - y^{(4)}(0)[a_1]$$

$$\Rightarrow L\{E_3\} = F(S)[7S^5 + 13S^4 + 19S^3 + 5S^2 + 2S + 6] - y(0)[7S^4 + 13S^3 + 19S^2 + 5S + 2]$$
$$- y'(0)[7S^3 + 13S^2 + 19S + 5] - y''(0)[7S^2 + 13S + 19] - y'''(0)[7S + 13]$$
$$- y^{(4)}(0)[7]$$

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